Abstract

The time-diversification effect has been widely analyzed at the individual level; many models show that young people should invest more in stocks than older people. In this paper, we study this effect at the aggregate level of a population; we show that it leads to a stock price decrease when the population is aging. This result casts doubt on the efficiency of time-diversification as a portfolio management strategy in the next years.

Time diversification, demographic structure and asset returns

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1 Introduction[1]

Time diversification is usually understood as a positive link between the proportion of wealth invested in stocks and the horizon of the investor. It is a usual advice given by fund managers to their clients, especially to those who save for retirement.

This advice is essentially based on two arguments ; the first one is the observation of long term returns on stocks during the twentieth century. For example, on US markets, the equity premium was around 6% on a yearly basis, as documented by Mehra and Prescott (1985) and Kocherlakota (1996) among others. The second reason often pointed out is a statistical property of sequences of independent random variables. The expectation of a sum increases linearly with the number of terms but the standard deviation only increases with the square root of the number of terms. Even if successive returns are not independent but exhibit mean reversion, the argument of time diversification is reinforced.

However, Samuelson (1963) was the first to address this problem and to point out that this way of reasoning is a fallacious interpretation of the law of large numbers. Using repeated lotteries, he showed that if an agent rejects a lottery at all wealth levels, he will also reject any sequence of i.i.d lotteries with the same distribution. Using the option pricing methodology, Bodie (1995) came to the same conclusion. Ross (1999), coming back on Samuelson's result , argued that the assumption " at all wealth levels " considerably restricts the utility functions concerned by this result. In fact, only CARA functions satisfy this assumption.

When considering the portfolio approach, the results are also controversial. Samuelson (1969) and Merton (1969), proved that the proportion invested in stocks is independent of the horizon when agents are characterized by CRRA utility functions and stock returns are driven by a brownian motion. But when future labor income is taken

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into account, the story changes ; for example, if the agents can borrow against this future income (assumed deterministic), the proportion invested in stocks increases with the horizon. Even with a random future labor income, time diversification is optimal as long as the correlation between labor income and stock returns is low (Viceira, 2001). Other more sophisticated models coming to the same conclusion are reviewed in Campbell and Viceira (2002).

All these models determine the optimal proportion of stocks to be held in a portfolio at the individual level. In this paper, we want to address a different, but closely related, question : if all people practice time diversification, what happens to stock prices when the population ages?

The question is important to understand what may happen in the next years when baby boomers will retire. A detailed empirical study of the link between demographic structure and asset returns was provided by Poterba (2001). No clear relationship appears between stock returns and demographic structure but it also brings to light that, at the aggregate level, time diversification is not the dominant strategy to manage portfolios. More precisely, the proportion of stocks with respect to the global wealth is not a monotonic function of age[2].

It should be noticed that historical data do not exhibit the kind of shift in the demographic structure we may expect for the next decades. Abel (2001), in the framework of a rational expectations model, showed that prices are likely to fall when the proportion of retired people grows ; Yoo (1994) simulated a baby boom and found that asset prices first increase and then decrease, the importance of the decline depending on whether or not the supply of capital is variable.

In this paper we specialize the analysis to prove that time diversification is not a good advice (as a general portfolio choice model) when the population is aging.

The paper is organized as follows. In section 2, we describe the economy and define the notations. Sections 3 and 4 are devoted to our main results and section 5 gives an illustration of our assumptions ; we use population projections provided by the US Census Bureau for the twenty first century.

Section 6 concludes the paper and proposes directions for further research.

2 Description of the economy

We consider a financial market with two traded assets; the risk-free asset generates a non stochastic return r per period and the risky asset (referred to as the "stock" in the following) is driven by a stochastic process characterized by an expected return equal to μ. A common assumption in financial models is to assume that the stock price is driven by a geometric brownian motion defined by :

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t$$

[2] See also Ameriks and Zeldes (2000).

where Z is a standard brownian motion. However, in the following, only the first moment will be important.

The population of agents is divided into T generations; we assume that in generation t, all agents have the same age a_t where $a_t \geq a_{t+1}$. In fact, the index t refers to the horizon of investment; then, older people have a shorter horizon which explains the ranking of ages. The vector of ages is denoted as A with $A' = (a_1, ..., a_T)$. The vector of proportions of agents in each generation is denoted as $\gamma' = (\gamma_1, ..., \gamma_T)$; we then have $\gamma' \mathbf{1} = 1$, where $\mathbf{1}' = (1, ..., 1)$ is a vector of $\mathbb{R}^T$ with all components equal to 1.

$\pi_t(\mu)$ is the proportion of stocks in the portfolio of generation-t agents when the equilibrium expected return on the stock is μ. $\pi_t(\mu)$ is an increasing function of μ. We denote $\Pi'(\mu) = (\pi_t(\mu), t = 1, ..., T)$

The time-diversification assumption is written as :

$$\pi_t(\mu) < \pi_{t+1}(\mu)$$

for every t and μ.

We first assume that per capita wealth is uniformly distributed across generations; this assumption is relaxed in section 4. The equilibrium condition in such a simple market is characterized by a constant value for $\sum_{t=1}^{T} \pi_t(\mu) \gamma_t$ since $\pi_t(\mu) \gamma_t$ is the proportion of stocks held by generation t. The equilibrium expected return then depends on the demographic structure γ. If γ moves to γ^*, μ moves to μ^*; the question addressed in this paper is to characterize the variations $\gamma^* - \gamma$ which lead to a increase in the expected equilibrium return on the stock, that is $\mu^* - \mu > 0$.

There are in fact two ways to account for changes in the population structure ; the first one is to consider constant proportions γ and to change the age vector A. However, it is usual, for example to characterize the demographic structure of a nation, to keep the classes of ages fixed and to change the proportions through time. For example, the US Census Bureau presents the population data within ranges of four years. Consequently, we will describe a change in population age by a modification of the vector γ, the vector A being constant.

In the following section, we analyze two measures of change of the demographic structure; the first one is an evolution of the mean age and the second one is a change in the cumulative distribution of ages in the sense of first-order stochastic dominance.

3 Expected stock returns and population aging

The first question which naturally arises is the following : two structures γ and γ^* being given, when can we say that population γ^* is older than population γ.

3.1 Expected stock returns and the mean-age

Definition 1 *A mean age operator is a linear form A defined on $\mathbb{R}^T$ by :*

$$\forall \gamma \in \mathbb{R}^T, A(\gamma) = \sum_{t=1}^{T} a_t \gamma_t$$

where $a_1 > a_2 > ... > a_T$.

As precised before, the age operator may be identified to the vector A of ages considered to classify the population under consideration; we will also denote $A'\gamma$ for $A(\gamma)$.

We first analyze the relationship between the mean age of the global population and the equilibrium expected return on the risky asset. To do so, we consider two populations γ and γ^* such that $A'(\gamma - \gamma^*) < 0$, that is to say, the mean age of population γ^* is greater than the one of population γ.

To simplify the notations, let us denote $\Delta\gamma = \gamma - \gamma^*$. The following lemma will be useful to prove the two following propositions.

Lemma 1 *Let B the $(T-1, T)$ array defined by :*

$$b_{ii} = 1; b_{i,i+1} = -1; b_{ij} = 0 \text{ if } j \neq i, i+1$$

The rows of B, denoted as $B_1, ..., B_{T-1}$ are linearly independent and there exist $(\alpha_1, ..., \alpha_{T-1})$ such that :

$$\Delta\gamma = \sum_{t=1}^{T-1} \alpha_t B_t$$

Proof of the lemma : The first part is in fact obvious ; B takes the following form :

$$B = \begin{bmatrix} 1 & -1 & 0 & .... & ... \\ 0 & 1 & -1 & 0 & ... \\ 0 & 0 & 1 & -1 & ... \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

If we add a row to B with all 0 components but the last one equal to 1, we obtain a triangular array with a determinant equal to 1, that is to say an array of full rank T. Consequently, the rank of B is $T-1$. Moreover, it is easy to remark that each row of B is orthogonal to the vector $\mathbf{1}$. But $\Delta\gamma$ is also orthogonal to $\mathbf{1}$ because γ and γ^* are vectors of proportions. In $\mathbb{R}^T$, $\mathbf{1}$ cannot be orthogonal to T linearly independent vectors, so we deduce that there exist $(\alpha_1, ..., \alpha_{T-1})$ such that :

$$\Delta\gamma = \sum_{t=1}^{T-1} \alpha_t B_t$$

To be consistent with other notations, the B_t are considered as column vectors, even if they denote the rows of B.

We can now prove our first proposition.

Proposition 2 *If time diversification applies and if there exists* $\beta \in \mathbb{R}^T$ *such that* $a_t - a_{t+1} = \beta (\pi_t(\mu) - \pi_{t+1}(\mu))$, *we then have :*

$$A'\Delta\gamma < 0 \Rightarrow \mu^* > \mu$$

Proof :

A has decreasing components, so $BA << 0$ and $B\Pi(\mu) >> 0$ where we write $x >> (<<) \, 0$ when the vector x has strictly positive (negative) components.

The assumption $A'\Delta\gamma < 0$ can be written as $\sum_{t=1}^{T-1} \alpha_t A' B_t < 0$ thanks to lemma 1 but the assumption of the proposition allows us to write :

$$\sum_{t=1}^{T-1} \alpha_t A' B_t = \beta \sum_{t=1}^{T-1} \alpha_t \Pi'(\mu) B_t$$

$\Pi(\mu)$ and A being ranked in opposite directions, we have $\beta < 0$ and then $\sum_{t=1}^{T-1} \alpha_t \Pi'(\mu) B_t = \Pi'(\mu)(\gamma - \gamma^*) > 0$. But in equilibrium we must have $\Pi'(\mu)\gamma = \Pi'(\mu^*)\gamma^*$; it implies that $\Pi(\mu^*) > \Pi(\mu)$ or $\mu^* > \mu$.

3.2 Expected returns and "distributional" aging

Although the mean age is probably the simplest synthetic measure of the age of a population, it is not sufficiently precise to get general results about the evolution of the equilibrium expected return on the risky asset when the population ages. In fact, the result of proposition 2 is based on a very strong assumption linking the demographic structure and the proportion invested in stocks in each generation. We consider now a more "complete" characterization of aging by means of a first order stochastic dominance criterion.

Definition 2 *Population* γ^* *is said older than population* γ *at the first order if and only if :*

$$\forall t \in \{1, .., T\}, \quad \sum_{s=t}^{T} \gamma_t^* \leq \sum_{s=t}^{T} \gamma_t$$

and there exists at least one t for which the inequality is strict.

Recall that index t denotes the horizon t so the inequality could be equivalently written as :

$$\sum_{s=1}^{t} \gamma_t^* \geq \sum_{s=1}^{t} \gamma_t$$

where each side gives the proportion of people older than a_t in each population. In other words, the cumulative distribution function of the random variable "age" is lower in population γ^*. This is exactly the definition of first-order stochastic dominance.

This definition allows us to prove our main result.

Proposition 3 *If γ^* is older than γ at the first order and if time diversification applies, the equilibrium expected return μ^* is greater than μ.*

Proof : Let C denote a (T,T) triangular array of the following form :

$$C = \begin{bmatrix} 1 & 1 & 1 & .... & ... \\ 0 & 1 & 1 & 1 & ... \\ 0 & 0 & 1 & 1 & ... \\ 0 & 0 & 0 & 1 & ... \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where the generic term C_{ts} is defined by $C_{ts} = 1$ if $s \geq t$ and $C_{ts} = 0$ if $s < t$. γ^* is older than γ if and only if[3] :

$$C\Delta\gamma > 0$$

Using $\Delta\gamma = \sum_{t=1}^{T-1} \alpha_t B_t$ we get :

$$\sum_{t=1}^{T-1} \alpha_t C B_t > 0 \tag{1}$$

But due to the definitions of C and the B_t, we can observe that :

$$\sum_{t=1}^{T-1} \alpha_t C B_t = \begin{bmatrix} 0 \\ -\alpha_1 \\ -\alpha_2 \\ ... \\ -\alpha_{T-1} \end{bmatrix}$$

Equation 1 implies that the vector $\alpha < 0$ where $\alpha' = (\alpha_1, ..., \alpha_{T-1})$; considering now $\Pi'(\mu)\Delta\gamma = \sum_{t=1}^{T-1} \alpha_t \Pi'(\mu) B_t$, we can deduce that $\Pi'(\mu)\Delta\gamma > 0$ because $\Pi'(\mu) B_t < 0$ by the time diversification assumption. The desired result follows, that is $\mu^* > \mu$ because $\Pi'(\mu)\gamma = \Pi'(\mu^*)\gamma^*$ in equilibrium.

[3]If x and y are two vectors in $\mathbb{R}^T$, we write $x > y$ if $x_t \geq y_t$ for every t and there exists at least one t such that $x_t > y_t$.

4 Non uniform distribution of wealth

Proposition 3 defines the conditions under which prices are likely to decrease when population is aging (in the distributional sense of definition 2). The result is obtained by assuming a uniform distribution of per capita wealth across generations. However, the data provided by Poterba (2001) and, more generally, by the Survey of Consumer Finances[4] do not confirm this assumption. Wealth is first increasing until retirement and then decreases sharply after the age of 65.

In this section, our aim is to find conditions on the (non-uniform) distribution of wealth that keep the preceding result valid.

let $Q_T = (q_1, ..., q_T)$ the vector of per capita wealths across generations ; the equilibrium condition when considering two populations γ and γ^* is now :

$$\sum_{t=1}^{T} \pi_t(\mu)\gamma_t q_t = \sum_{t=1}^{T} \pi_t(\mu^*)\gamma_t^* q_t$$

If γ^* is older than γ in the sense of definition 2, we get $\mu^* > \mu$ if :

$$\sum_{t=1}^{T} \pi_t(\mu)(\gamma_t - \gamma_t^*) q_t > 0$$

Using the notation $\Delta\gamma_t$ defined in the preceding section, this inequality can be rewritten as :

$$\sum_{t=1}^{T} \pi_t(\mu) q_t \Delta\gamma_t > 0 \tag{2}$$

Let us denote $\pi_t(\mu)q_t = \theta_t(\mu)$; equation 2 leads to the following result.

Proposition 4 *If* $\theta_1(\mu) \leq \theta_2(\mu..... \leq \theta_T(\mu)$ *and* γ^* *is older than* γ, *time-diversification implies* $\mu^* > \mu$.

Proof : The first part of the proof of proposition 3 can still be used to write :

$$\sum_{t=1}^{T-1} \alpha_t C B_t = \begin{bmatrix} 0 \\ -\alpha_1 \\ -\alpha_2 \\ ... \\ -\alpha_{T-1} \end{bmatrix} > 0$$

implying $\alpha < 0$.

But $\theta'\Delta\gamma = \sum_{t=1}^{T-1} \alpha_t \theta' B_t$ leads to $\theta'\Delta\gamma > 0$ because the assumption $\theta_1 \leq \theta_2..... \leq \theta_T$ means $\theta' B_t < 0$.

It then follows that $\mu^* > \mu$ since $\theta'(\mu).\gamma = \theta'(\mu^*)\gamma^*$ in equilibrium.

[4] see Kennikel *and al.* (2000) for the detailed report.

Corollary 5 *If per capita wealth is a decreasing function of age, time-diversification implies $\mu^* > \mu$.*

We know that time-diversification means $\pi_t(\mu) \leq \pi_{t+1}(\mu)$; if $q_1 \leq q_2 ... \leq q_T$ we get obviously that $\pi_t(\mu)q_t \leq \pi_{t+}(\mu)q_{t+1}$ that is $\theta_t(\mu) \leq \theta_{t+1}(\mu)$ and proposition 4 gives the desired result.

5 Illustration

The result proved in proposition 3 seem to be based on a very strong assumption, that is the first-order stochastic dominance criterion. It is then important to evaluate if this hypothesis is really restrictive or irrealistic. To manage this task, we consider population projections provided by the U.S Census Bureau for the 21st century. These projections are available on the web site of the Census Bureau[5]. Four series of projections can be found which differ by the assumptions on fertility, mortality and immigration. For our illustration, we chose to consider what is called "Middle Series"; there are also "Low Series", "High Series" and "Zero International Migration Series". These projections are based on 1998 data of the Bureau of Census. All the details concerning the cohort-component method used to obtain these estimations may be found in Hollman *et al.* (1998).

FIGURE 1 around here

The series give the number of people of each age between 0 and 100 for the years 1999 to 2100. To illustrate our result, we compare the populations in 2000, 2005 and 2010. Let F_1, F_2 and F_3 the three cumulative distribution functions of ages for the three years. Figure 1 represents the differences $F_1 - F_2$ (the solid line) and $F_1 - F_3$ (the dashed line). We can observe that the first-order stochastic dominance assumption is "almost" verified, that is to say, the two differences are always negative, except for the range 65-70 on the first curve.

This means that our criterion of stochastic dominance is a realistic way to characterize the aging of a population. However, when a non-uniform distribution of wealth is considered, the result is more restrictive since the condition proposed in proposition 4 is only a sufficient one (wealth invested in stock decreases with age). The data provided by the Survey of Consumer Finances show that per capita wealth is inverse U-shaped with young and old people "poorer" than middle-aged people. The condition of proposition 4 is then not satisfied.

6 Concluding remarks

In this paper, we showed that when the population ages, the equilibrium expected return on stocks is likely to increase if investors time diversify when wealth is uni-

[5] www.census.gov/population/www/projections/natdet-D1A.html

formly distributed across generations ; in other words, prices are likely to fall in the next years if agents follow the usual advice provided by most financial advisers.

Two sets of assumptions were considered, depending on the notion of "population age ". When only the mean age is considered, the result needs a strong relationship between the age distribution in the population and the proportion of stocks held in portfolios. It is not surprising because the mean age is a very rough measure to characterize the aging of a population. When aging is characterized by means of stochastic dominance, the result comes from the only assumption of time-diversification in populations. Even if first order stochastic dominance may appear as a strong criterion, it is almost satisfied by population projections in the US for the next decades. Similar data for european countries[6] lead to the same observation. Considering non uniform distributions, we found a sufficient condition to keep the preceding result valid. However, this condition is not met by empirical data when usual categories of age are considered, as in the Survey of Consumer Finances. Further research has then to be done at several levels; first, it is necessary to link in a more precise way demographic evolution, wealth repartition between generations and, last but not least, changes in risk attitudes across generations. Second, birth and death processes have to be used to characterize the dynamics of populations; in fact, in this paper, we do not analyze how the population evolves from γ to γ^*.

Finally, the conditions provided here are sufficient and can surely be weakened. Moreover, implications about the structure of the vector π must be studied to provide policy implications; in fact, it is difficult to change the evolution of the age structure but changing the structure of portfolios across generations may be easier.

[6] The US Census Bureau provides projection data for most countries in 2025. We have verified the stochastic dominance criterion for UK, Germany, France and Italy by comparing years 2000 and 2025.

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Figure 1: Changes in age distribution function