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MULTIPLICATIVE BACKGROUND RISK *

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Abstract

Although there has been much attention in recent years on the effects of additive background risks, the same is not true for its multiplicative counterpart. We consider random wealth of the multiplicative form $\tilde{x}\tilde{y}$, where $\tilde{x}$ and $\tilde{y}$ are statistically independent random variables. We assume that $\tilde{x}$ is endogenous to the economic agent, but that $\tilde{y}$ is an exogenous and nontradable background risk, which represents a type of market incompleteness. Our main focus is on how the presence of the multiplicative background risk $\tilde{y}$ affects risk-taking behavior for decisions on the choice of $\tilde{x}$. We characterize conditions on preferences that lead to more cautious behavior.

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1. Introduction

Consider a risk-averse economic agent whose preferences can be represented within an expected-utility framework via the continuously differentiable utility function u. The agent must decide upon choice parameters for a random variable representing final wealth, $\tilde{x}$. For example, $\tilde{x}$ might represent wealth from an individual's portfolio of financial assets, or $\tilde{x}$ might represent random corporate profits based on management decisions within the firm.

A fair amount of attention in recent years has examined how decisions on $\tilde{x}$ might be affected by the addition of an additive risk $\tilde{\varepsilon}$, where $\tilde{\varepsilon}$ and $\tilde{x}$ are statistically independent. Thus, final wealth or profits can be written as $\tilde{x}+\tilde{\varepsilon}$. The market is assumed to be incomplete in that $\tilde{\varepsilon}$ is not directly insurable. For example, $\tilde{\varepsilon}$ might represent future wage income subject to human-capital risks; or $\tilde{\varepsilon}$ might represent an exogenous pension portfolio provided by one's employer. Although it is interesting to examine the interdependence between $\tilde{x}$ and $\tilde{\varepsilon}$, the case of independence is of special interest and provides for many interesting observations. In order to focus on the risk effects, rather than wealth effects, it is often assumed that $E\tilde{\varepsilon}=0$, where E denotes the expectation operator. In such a case, $\tilde{\varepsilon}$ is often called a "background risk." Since any non-zero mean for $\tilde{\varepsilon}$ can be added to the $\tilde{x}$ term, this assumption does not reduce the applicability of the model. Our purpose in the present paper is to examine the effects of introducing a "multiplicative background risk" into the individual's final wealth distribution.

The modern literature on additive background risk stems from the papers of Kihlstrom, et al. (1981), Ross (1981) and Nachman (1982). These papers focus on interpersonal behavior comparisons, mainly addressing the question: "If I am willing to pay more than you to rid myself of any fair lottery, would I still be willing to do so in the presence of an additive background risk?" Doherty and Schlesinger (1983) incorporated the analysis into intrapersonal models of decision making under uncertainty, focusing on differences in optimal behavior with vs. without a background risk. The literature underwent somewhat of a renaissance in the 1990's thanks to new theoretical tools provided by Pratt and Zeckhauser (1987), Kimball (1990) and Gollier and Pratt (1996).

One canonical hypothesis concerning additive background risk is that the riskiness of $\tilde{\varepsilon}$ leads to a more cautious behavior towards decisions on $\tilde{x}$. For example, Guiso, et al. (1996) use Italian survey data to show that individuals with a riskier perception of their (exogenously managed) pension wealth react by investing relatively more in bonds in their personal accounts. However, this conclusion need not always be the case in theory, unless particular restrictions on preferences are met. Eeckhoudt and Kimball (1992) first examined this direction of research. Rather than review the large body literature for the case of additive background risks, we refer the reader to the excellent comprehensive presentation of this material in Gollier (2001).

Surprisingly, very little attention has been given to the case where the background risk is multiplicative. Indeed, if one were to ask the reader to think of possible types of background risks, we believe that examples with multiplicative types of background risk would be at least as prevalent as additive ones. Our goal in this paper is to provide a theoretical foundation for models with a multiplicative background risk. Under what conditions on preferences will the presence of a multiplicative background risk compel the agent to behave more cautiously in making decisions about the endogenous wealth variable $\tilde{x}$?

To this end, let $\tilde{y}$ be a random variable on a positive support that is statistically independent of $\tilde{x}$. We consider final wealth to be given by the product $\tilde{x}\tilde{y}$. The random variable $\tilde{y}$ is considered to be exogenous to the individual and is not insurable. Numerous examples of such multiplicative risks include the following:

1. Let $\tilde{x}$ be the pre-tax profits of a firm and let $\tilde{y}$ represent the firm's retention rate net of taxes, where tax rates are random due to tax-legislation uncertainty.

2. Let $\tilde{x}$ be the random wealth in an individual's financial portfolio in period one, and let $\tilde{y}$ denote the return on a mandatory (and exogenously managed) annuity account that uses proceeds from $\tilde{x}$ in period two.

3. Let $\tilde{x}$ denote nominal wealth or profit and let $\tilde{y}$ denote an end-of-period price deflator.

4. Let $\tilde{x}$ denote profit in some foreign currency for which forward contracts or options are not available and let $\tilde{y}$ denote the end-of-period exchange rate.

5. Let $\tilde{x}$ denote the random per-unit profit for a farm commodity and let $\tilde{y}$ denote an exogenous random quantity of output.

In order to isolate the risk effects of $\tilde{y}$, we will assume that $E\tilde{y}=1$ throughout this paper. For the case where $\tilde{y}$ has a mean that differs from one, we can incorporate this mean into $\tilde{x}$ via a deterministic scaling effect.[1] Since $\tilde{x}\tilde{y} = \tilde{x} + \tilde{x}(\tilde{y}-1)$, the assumption that $E\tilde{y}=1$, together with the independence of $\tilde{x}$ and $\tilde{y}$, guarantees that $\tilde{x}\tilde{y}$ is riskier than $\tilde{x}$ alone in the sense of Rothschild and Stiglitz (1970). We will refer to $\tilde{y}$, defined in this manner with $E\tilde{y}=1$, as a "multiplicative background risk."

We should point out at the outset that the results for the multiplicative case do not simply mirror those of the additive case. For instance, consider a simple portfolio example with an allocative choice between risky stocks and risk-free bonds. The

[1] Thus, for instance, in our first example above we can let $\tilde{x}$ represent after-tax profits based on the expected tax rates and let $\tilde{y}$ represent a deviation from the expected after-tax retention rates. Or, in the second example let $\tilde{x}$ denote wealth including expected annuity returns and let $\tilde{y}$ denote a multiplicative excess-return adjustment.

individual has an initial wealth of 100 and the risk-free rate is assumed to be $r_f = 0.05$. The return on the stock portfolio is assumed to be log-binomial with an expected return of $E\tilde{r} = 0.11$ and a standard deviation of $\sigma = 0.20$ (implying that, in a binomial model, stocks either return about 33% or lose about 10%, each with an equally likely chance). Utility is assumed to belong to the HARA class with $u(x) = -\frac{1}{2}(x+a)^{-2}$, where a is a constant chosen such that $x+a$ remains positive over relevant wealth levels.[2] We note that preferences satisfy decreasing absolute risk aversion (DARA) for any choice of *a,* whereas relative risk aversion will be increasing [decreasing, constant] whenever a is positive [negative, zero]. We examine the addition of two alternative sources of background risk. The first is an additive background risk, for which final wealth is either increased or decreased by 30, each with probability one-half. The second is a multiplicative background risk, for which terminal wealth is either increased or decreased by 30 percent, each with a probability one-half. The optimal portfolio choices are illustrated in the following table.

TABLE 1: **Bond Proportions: Multiplicative vs. Additive Background Risk**

(All utility is DARA within the HARA class, $u(x) = -\frac{1}{2}(x+a)^{-2}$, initial wealth = 100)

(Relative risk aversion is constant for a=0, increasing for a=+25 and decreasing for a=-25)

Utility	Background Risk	Proportion in Bonds
$a = 0$	None	55%
	Additive	66%
	Multiplicative	55%
$a = +25$	None	45%
	Additive	54%
	Multiplicative	41%
$a = -25$	None	66%
	Additive	78%
	Multiplicative	70%

[2] Utility belongs to the HARA class if the measure of risk tolerance, $-u'(x)/u''(x)$, is linear in *x*.

In each case in the above example, the proportion of wealth invested in risk-free bonds increases when an additive background risk is included.[3] Since DARA inside of the HARA class of preferences also implies standard risk aversion (Kimball 1993), we know that bond proportions will always increase with an additive background risk. However, as the example shows, a multiplicative background risk might cause the bond proportion to shrink. In particular, when a = 25, so that we have both DARA and increasing relative risk aversion – hardly considered unusual cases – we then have a lower proportion of wealth invested in the risk-free bond. That is, the investor reacts to the multiplicative background risk by taking a more aggressive position in stocks.

Our paper will show how each of the situations in the example above can be determined qualitatively (i.e. whether more or fewer bonds are purchased in the presence of a background risk) before calculating the optimal portfolios. The fact that the qualitative effects might be predetermined by the parameters of the model implies that care must be taken when modeling various economic and/or financial phenomena. For example, seemingly innocuous assumptions made about preferences might actually predispose a model to achieve particular results.

We begin in the next section by introducing the basic framework. We next examine conditions on preferences that lead to more (or less) cautious behavior towards $\tilde{x}$ in the presence of a multiplicative background risk $\tilde{y}$. In section 4, we introduce the concept of the affiliated utility function and examine some of its basic properties. Section 5 uses affiliated utility functions to apply several extant results from the literature on additive background risks to the case of multiplicative background risks. Section 6 examines comparative risk aversion; in particular we determine conditions that preserve

[3] Note that, even for the cases with no background risk, since relative risk aversion is decreasing in a, we have the bond proportion falls as a rises. Our point in the table, however, is to compare the levels of bonds between various types of background risk for a fixed value of a.

the relation "more risk averse" in the presence of a multiplicative background risk. Section 7 provides some concluding thoughts.

2. The Basic Model

Consider a risk-averse economic agent with utility function u. We wish to determine how the addition of a multiplicative background risk $\tilde{y}$ affects decision making on $\tilde{x}$. Both $\tilde{x}$ and $\tilde{y}$ are assumed to be strictly positive a.s. Let F and G denote the (cumulative) distribution functions associated with the random variables $\tilde{x}$ and $\tilde{y}$ respectively. Since $\tilde{x}$ and $\tilde{y}$ are independent, we can write expected utility as the iterated integral

$$(1) \qquad Eu(\tilde{x}\tilde{y}) = \int_0^\infty\int_0^\infty u(xy)dG(y)dF(x) \equiv E_F[E_G u(\tilde{x}\tilde{y})].$$

Define the derived utility function, see Nachman (1982)[4], as the interior integral given in equation (1). That is,

$$(2) \qquad v_G(x) \equiv \int_0^\infty u(xy)dG(y) = E_G u(x\tilde{y})$$

Trivially, $v_G(x)$ is increasing and concave since u is. Thus, equation (1) can be written as $Eu(\tilde{x}\tilde{y}) = E_F v_G(\tilde{x})$. Decisions on $\tilde{x}$ made in the presence of the multiplicative risk $\tilde{y}$ under utility u are isomorphic to decisions made on $\tilde{x}$ in isolation under the risk-averse utility $v_G(x)$. Let $\Gamma(\tilde{x})$ denote the set of positive random variables $\tilde{y}$ such that $\tilde{y}$ is statistically independent from $\tilde{x}$ and $E\tilde{y}=1$. Our focus here is in determining conditions

[4] Actually, Nachman considers a more general relationship between $\tilde{x}$ and $\tilde{y}$. We adapt his measure to the case of multiplicative risks. The derived utility function for the additive case is described earlier by Kihlstrom, et al. (1981).

on the utility function u such that the derived utility function, $v_G(x)$, is more risk averse than u for all $\tilde{y} \in \Gamma(\tilde{x})$. In other words, we wish to determine conditions on u that will guarantee that

$$(3) \qquad \frac{-v''_G(x)}{v'_G(x)} \equiv \frac{-E_G[u''(x\tilde{y})\tilde{y}^2]}{E_G[u'(x\tilde{y})\tilde{y}]} \geq \frac{-u''(x)}{u'(x)} \qquad \forall x.$$ [5]

To avoid excessive notation, we will dispense with the subscripts and simply write $v(x)$ and $Eu(x\tilde{y})$, where we assume $\tilde{y}$ is an arbitrary member of $\Gamma(\tilde{x})$. We will let $r_v(x)$ and $r_u(x)$ denote the measure of absolute risk aversion for v and u respectively, i.e. the left-hand-side and right-hand-side of inequality (3) respectively.

Since we are involved with a multiplicative background risk, it is often convenient to consider the corresponding measures of relative risk aversion, $R_v(x) \equiv xr_v(x)$ and $R_u(x) \equiv xr_u(x)$. Obviously, for any positive wealth level x, $r_v(x) \geq r_u(x)$ if and only if $R_v(x) \geq R_u(x)$.

For arbitrary x, straightforward manipulation of (3) shows that

$$(4) \qquad R_v(x) = E[R_u(x\tilde{y})\frac{u'(x\tilde{y})\tilde{y}}{E[u'(x\tilde{y})\tilde{y}]}] \equiv \int_0^\infty R_u(xy)d\eta_x(y)$$

where $\eta_x(y) \equiv \int_0^y \frac{u'(xt)tdG(t)}{E_G[u'(x\tilde{y})\tilde{y}]}$.

Note that $\eta_x(y)$ is itself a well-defined probability distribution. We define $\hat{E}_x$ to denote the expectation operator based on the probability distribution $\eta_x(y)$, which is a

[5] In order to keep the mathematics simple, we will take "more risk averse" to be in the weak sense of Pratt (1964).

type of risk-adjusted probability measure.[6] Thus, we see that relative risk aversion for *v* is a weighted average of relative risk aversion for *u*, namely $R_v(x) = \hat{E}[R_u(x\tilde{y})]$.

3. Risk Aversion Properties

From equation (4), it follows trivially that *v* inherits constant relative aversion (CRRA), whenever *u* exhibits CRRA. More explicitly, if $R_u(x) = \gamma \quad \forall x$, then $R_v(x) = \gamma \;\; \forall x$ as well. Since it then also follows that $r_u(x) = r_v(x) \;\; \forall x$, we see that *u* and *v* are equivalent utility representations under CRRA. This is not surprising, since any optimal choice of an endogenous $\tilde{x}$ also will be optimal for $\tilde{x}y$, for every constant positive level of *y* under CRRA preferences.

We next wish to examine conditions under which (3) holds $\forall \tilde{y} \in \Gamma(\tilde{x})$, i.e., we want to know when *v* is more risk averse than *u*. We may consider conditions for which this holds locally, with $r_v(x) \geq r_u(x)$, by examining the equivalent condition $R_v(x) \geq R_u(x)$. Our approach is to consider this last inequality for a particular value of *x*, by applying η_x as in equation (4). If the value of *x* chosen is arbitrary, so that $R_v(x) \geq R_u(x) \;\; \forall x$, then we are done.

Suppose that $R_u(x)$ is (not necessarily strictly) convex. Since $\eta_x(y)$ is a probability distribution, it follows from Jensen's inequality and equation (4) that

(5) $$R_v(x) \equiv \hat{E}R_u(x\tilde{y}) \geq R_u(x\hat{E}\tilde{y}),$$

where

[6] If we have a representative agent model, and if we confine ourselves to a fixed value of *x*, this measure is simply the "risk-neutral probability measure." The random variable $[u'(x\tilde{y})\tilde{y}]/E[u'(x\tilde{y})\tilde{y}]$ in equation (4) is the Radon-Nikodym derivative of this measure with respect to *G*, again conditional on a fixed value of *x*.

(6) $$\hat{E}\tilde{y} = \int_0^\infty y d\eta_x(y) = \int_0^\infty y \frac{u'(xy)y}{E[u'(x\tilde{y})\tilde{y}]} dG(y).$$

Next, note that

(7) $$\frac{\partial^2 u(xy)}{\partial x \partial y} = \frac{\partial}{\partial y}[u'(xy)y] = u'(xy)[1 - R_u(xy)].$$

The sign of (7) tells us whether increases in the level of *y* will increase or decrease the marginal utility of *x*. The derivative in (7) will be everywhere positive [negative] if $R_u(xy) < [>] 1 \ \forall y$ in the support of *G*. This implies that increases in *y* reduce the marginal utility of *x* whenever $R_u > 1$, and increases in *y* increase the marginal utility of *x* whenever $R_u < 1$.

Since $E\left\{\frac{u'(x\tilde{y})\tilde{y}}{E[u'(x\tilde{y})\tilde{y}]}\right\} = 1$, we see from (6) and (7), for example, that $R_u > 1$ everywhere implies that the probability measure $\eta_x(y)$ puts relatively more weight on lower values of *y* than does the true probability measure *G*(*y*). The opposite is true if $R_u < 1$. We thus obtain the following result from (6) and (7).

<u>*Lemma 1*</u>: $\hat{E}\tilde{y} \gtreqless E\tilde{y} = 1$ if $R_u(xy) \lesseqgtr 1 \quad \forall y \in Supp(G)$.

We are now ready to prove the following result:

<u>*Proposition 1*</u>: *Suppose that $R_u(x)$ is convex and that one of the following conditions holds $\forall (x, y) \in Supp(F) \times Supp(G)$:*

(i) $R_u(xy) > 1$ and R_u is decreasing,

or (ii) $R_u(xy) < 1$ and R_u is increasing.

Then v is more risk averse than u.

Proof: Since $R_u(x)$ is convex, it follows from equation (4) that $R_v(x) \geq R_u(x\hat{E}\tilde{y})$ by Jensen's inequality. If $R_u > 1$, then $\hat{E}\tilde{y} < 1$ from Lemma 1. Hence, $R_u(x\hat{E}\tilde{y}) \geq R_u(x)$ under the assumption of decreasing relative risk aversion (DRRA). If $R_u < 1$, then it follows from Lemma 1 that $\hat{E}\tilde{y} > 1$. Hence, $R_u(x\hat{E}\tilde{y}) \geq R_u(x)$ under the assumption of increasing relative risk aversion (IRRA). Thus we have $R_v(x) \geq R_u(x)$ whenever condition (*i*) or (*ii*) holds. ■

Interestingly, if we have CRRA preferences, we have already seen that *u* and *v* are equivalent regardless of whether or not relative risk aversion exceeds one. If relative risk aversion is increasing in wealth, as originally postulated by Arrow (1971) and empirically supported by much literature, most recently by Guiso and Paiella (2001), then *v* will be more risk averse than *u* whenever R_u is convex and less than 1. If R_u is everywhere greater than 1 and exhibits increasing relative risk aversion, we cannot use Proposition 1 to verify that *v* is more risk averse than *u*. Indeed, if we have $R_u > 1$ and if R_u is (not necessarily strictly) concave, it is easy to show that *v* is then less risk averse than *u*. Indeed, the following two cases are easy to show.

Proposition 2: *Suppose that $R_u(x)$ is concave and that one of the following conditions holds $\forall (x, y) \in Supp(F) \times Supp(G)$:*

(i) $R_u(xy) > 1$ and R_u is increasing,

or (ii) $R_u(xy) < 1$ and R_u is decreasing.

Then v is less risk averse than u.

Proof: The proof is similar to Proposition 1 and left to the reader. ■

Of course, whether risk aversion exhibits constant-, increasing-, or decreasing relative risk aversion, or none of these, is an empirical question. Certainly constant relative risk aversion is very common in equilibrium asset-pricing models. But empirical support also exists for both increasing relative risk aversion (e.g. Guiso and Paiella (2001)) and for decreasing relative risk aversion (e.g. Ogaki and Zhang (2001)). Whether relative risk aversion might be concave or convex in wealth has not received much attention at all until fairly recently. For example, Aït-Sahalia and Lo (2000) examine S&P 500 option prices to find some evidence of an oscillating level of relative risk aversion, although they do find *R* to be decreasing and convex at relatively low levels of wealth.[7] Aït-Sahalia and Lo (2000) also review much of the literature examining whether relative risk aversion is greater- or less-than one, with most support these days finding $R>1$.

To illustrate Proposition 1 and 2, consider the following examples:

Example 1: Let $u(x) = -e^{-kx}$ where $k > 0$. This is the case of constant absolute risk aversion (CARA). In this case $R_u'(x) = k$ and $R_u''(x) = 0$. Thus, R_u is increasing and is both convex and concave. If we consider $\tilde{x}$ and $\tilde{y}$ such that $xy < 1/k$ $\forall(x, y) \in \text{Supp}(F) \times \text{Supp}(G)$, then $R_u(xy) < 1$ and v is more risk averse than *u* by Proposition 1. However, if $xy > 1/k$ $\forall(x, y) \in \text{Supp}(F) \times \text{Supp}(G)$, then $R_u(xy) > 1$ and v is less risk averse than *u* by Proposition 2.

Example 2: Let $u(x) = x - kx^2$ where $k > 0$. We restrict $x < \frac{1}{2k}$ so that marginal utility is positive. This is the case of quadratic utility. It is straightforward to show that $R_u(x) = 2kx(1-2kx)^{-1}$ and that R_u is both strictly increasing and convex. Moreover, $R_u(xy) < 1$ if $xy < \frac{1}{4k}$ $\forall(x, y) \in \text{Supp}(F) \times \text{Supp}(G)$, so that v is more risk averse than u by Proposition 1. In other words, v is more risk averse than u over the first half of the relevant (upward-sloping) range of the quadratic utility function. On the other

7 See also Jackwerth (2000).

hand, if $\frac{1}{4k} < xy < \frac{1}{2k} \quad \forall (x,y) \in \text{Supp}(F) \times \text{Supp}(G)$, then $R_u(xy) > 1$, but we cannot apply Proposition 1 (since R_u is increasing) or Proposition 2 (since R_u is convex).

Both utility functions above belong to the HARA class of utility, as does CRRA utility. Since we already showed that u and v are equivalent under CRRA, we see that no general results seem to apply to the HARA class of utility. However, we have more tractability in the shape of R_u under HARA. Let $u(x) = \xi(\eta + \frac{x}{\gamma})^{1-\gamma}$, where $\eta + \frac{x}{\gamma} > 0$ and $\frac{\xi(1-\gamma)}{\gamma} > 0$. Straightforward calculations show that $R_u'(x) = \eta(\eta + \frac{x}{\gamma})^{-2}$ and that $R_u''(x) = -[\frac{2}{\gamma}(\eta + \frac{x}{\gamma})^{-1}]R_u'(x)$. Thus, for the case of constant absolute risk aversion ($\gamma \to \infty$), we obtain $R_u'(x) = k$ and $R_u''(x) = 0$, as in Example 1. If we have increasing absolute risk aversion, then we must have $\gamma < 0$ and $\eta > 0$. It follows that $R_u'(x) > 0$ and $R_u''(x) > 0$, so that we must have R_u increasing and convex, as is the case with quadratic utility in our Example 2.

If we have the fairly common case of decreasing absolute risk aversion (DARA) within the HARA class of utility, then $\gamma > 0$. Hence, $\text{sgn}\, R_u''(x) = -\text{sgn}\, R_u'(x)$. Consequently, we must have R_u either (*i*) constant, (*ii*) decreasing and convex, or (*iii*) increasing and concave. In particular, corresponding to cases (*i*) - (*iii*) above:

(i) If u satisfies CRRA, then v and u are equivalent.

(ii) If $R_u > 1$, which is the case in our example in the introduction of this paper (see Table 1), as well as decreasing and convex, then v is more risk averse than u by Proposition 1.

(iii) If $R_u > 1$, as well as increasing and concave, then v is less averse than u by Proposition 2.

Thus, if preferences are DARA, it follows that we might have v either more risk averse than u, less risk averse than u or equally as risk-averse as u.

4. Affiliated Utility Functions

In this section, we obtain additional results by considering $\ln(xy) = \ln x + \ln y$. This allows us to adapt several results from the case of additive background risks to the multiplicative case. In order to accomplish this, we define the affiliated utility function, $\hat{u}$, such that $u'(x) = \hat{u}'(\ln x)$, where we restrict $x > 0$. Equivalently, we can substitute $\theta = \ln x$ to define $\hat{u}'(\theta) \equiv u'(e^{\theta}) \quad \forall \theta \in \mathbb{R}$. In other words, $\hat{u}'$ is the composite of *u'* with the exponential function. Since $u'(xy) = \hat{u}'(\ln x + \ln y)$, we will examine the additive risks $\ln \tilde{x} + \ln \tilde{y}$ in this section.[8]

In order to obtain $\hat{u}'$ as defined above, we define the affiliated utility function $\hat{u}$ as follows:

$$(8) \qquad \hat{u}(\ln x) \equiv \int x^{-1} du(x) = \int \frac{u'(x)}{x} dx \, .$$

Note that the definition in (8) implies that:

$$(9) \qquad \begin{aligned} &\hat{u}'(\ln x) = u'(x) \\ &\hat{u}''(\ln x) = xu''(x) \\ &\hat{u}'''(\ln x) = xu''(x) + x^2 u'''(x). \end{aligned}$$

From (9), we see that $\hat{u}$ will be both increasing and concave, whenever u is. In other words, $\hat{u}$ is itself has the properties of a well-defined risk-averse utility function.[9]

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[8] The reason for defining $u'(x) = \hat{u}'(\ln x)$, rather than $u(x) = \hat{u}(\ln x)$, is that we obtain stronger results.

[9] We caution however that $\hat{u}$ is only a useful device and does not represent anyone's utility directly.

Although the definition of affiliated utility in (8) might seem a bit strange, it is essentially the derivatives in (9) that interest us. The function $\hat{u}$ is well defined for positive x. For example, consider the following three common utility functions:

Example 3:

(a) Let $u(x) = \frac{1}{1-\gamma} x^{1-\gamma}$. This is CRRA with relative risk aversion $R = \gamma$.

In this case, we define

$$\hat{u}(\ln x) = \int \frac{u'(x)}{x} dx = \int \frac{x^{-\gamma}}{x} dx = -\tfrac{1}{\gamma} x^{-\gamma}.$$

Thus,

$$\hat{u}'(\ln x) = x^{-\gamma} = u'(x).$$

(b) Let $u(x) = -\frac{1}{a} e^{-ax}$, which is CARA with absolute risk aversion $r = a$.

Now define $\hat{u}$ via the power series

$$\hat{u}(\ln x) = \int \frac{e^{-ax}}{x} dx = \ln x - ax + \frac{(-ax)^2}{2 \cdot 2!} + \frac{(-ax)^3}{3 \cdot 3!} + \dots$$

which implies that

$$\hat{u}'(\ln x) = e^{-ax} = u'(x).$$

(c) Let $u(x) = x - kx^2$ for $x < \frac{1}{2k}$, which is quadratic utility.

For quadratic utility we define

$$\hat{u}(\ln x) = \int \frac{1-2kx}{x} dx = \ln x - 2kx$$

so that

$$\hat{u}'(\ln x) = 1 - 2kx = u'(x).$$

Let $\hat{r}(\theta)$ denote the measure of absolute risk aversion for $\hat{u}(\theta)$, i.e. $\hat{r}(\theta) = -\hat{u}''(\theta)/\hat{u}'(\theta)$. Straightforward calculations show that

$$(10) \qquad R_u(x) = -\frac{\hat{u}''(\ln x)}{\hat{u}'(\ln x)} = \hat{r}_u(\ln x)\,.$$

Similarly using the definition of $v(x)$ in (2), define $\hat{v}(\ln x)$ such that $\hat{v}'(\ln x) = v'(x)$. Then, in a manner analogous to equation (10) we can further derive

$$(11) \qquad R_v(x) = -\frac{E\hat{u}''(\ln x + \ln \tilde{y})}{E\hat{u}'(\ln x + \ln \tilde{y})} \equiv \hat{r}_v(\ln x)\,.$$

From (10) and (11), we easily observe the following result.

<u>Lemma 2</u>: *(i)* $R_v(x) \geq R_u(x)$ *if and only if* $\hat{r}_v(\ln x) \geq \hat{r}_u(\ln x)$,

and *(ii)* $R_t(x)$ *is decreasing if and only if* $\hat{r}_t(\ln x)$ *is decreasing* , *t = u, v.*

Note that, equivalent to (*i*) above, we also can write $r_v(x) \geq r_u(x)$ if and only if $\hat{r}_v(\ln x) \geq \hat{r}_u(\ln x)$.

Consider now the set of $\tilde{y} \in \Gamma(\tilde{x})$, so that $E\tilde{y} = 1$. For any $\tilde{y} \in \Gamma(\tilde{x})$, $E(\ln \tilde{y}) \leq 0$, with equality only holding in the degenerate case, where $\tilde{y} = 1$ a.s. We know from Gollier and Pratt (1996) that the addition of a nonpositive-mean, <u>additive</u> background risk will always make the derived utility function more risk averse if and only if utility is risk vulnerable. Applying the Gollier and Pratt result to $\hat{u}$, it follows that v is more risk averse than u for every <u>multiplicative</u> background risk $\tilde{y} \in \Gamma(\tilde{x})$ if and only if $\hat{u}(x)$ is risk vulnerable.[10] Since risk vulnerability is not an easy trait to verify, Gollier and Pratt offer us several useful sufficient conditions for risk vulnerability that are easy to check. In particular, we can apply their results and Lemma 2(*i*) to obtain the following result.

<u>Proposition 3</u>: *The derived utility v is more risk averse than u if either*

(*i*) $\hat{r}_u$ *is decreasing and convex,*

or (*ii*) $\hat{u}$ *exhibits standard risk aversion (see Kimball, 1993, and below).*

In some instances, we might be able to check the conditions on $\hat{u}$ in Proposition 3 directly. For instance, using our example from Table 1 in the introduction, we had $u(x) = -\frac{1}{2}(x+a)^{-2}$. Consider the case where $a<0$ and where we restrict $x+a>0$. In our example we saw that *v* was more risk averse than *u*. We also saw in the previous section how this followed from Proposition 1(*i*). This result also follows directly from Proposition 3(*i*). To see this, define $\hat{u}'(\ln x) = u'(x) = (x+a)^{-3}$. Straightforward calculations show that $\hat{r}_u$ is decreasing and convex, when $a<0$. Proposition 1(*i*) requires that $R_u > 1$, which holds in this example. Indeed $R_u > 3$ when $a < 0$ for all *x* with this utility function. However, $R_u > 1$ is not required for Proposition 3 to hold. For example, if we define $u(x) = 2(x+a)^{1/2}$ with $a < 0$, then Proposition 3(*i*) will apply.

5. Properties of Utility and its Affiliate

In this section, we examine some conditions on the utility function $u(x)$ that must hold if its affiliated utility function $\hat{u}(\theta)$ satisfies the properties given in Proposition 3(*i*)

[10] More directly, we would use Gollier and Pratt to examine the behavior of $\hat{v}(\ln x) \equiv E\hat{u}(\ln x + \ln \tilde{y})$ for any nondegenerate $\tilde{y}$ with $E \ln \tilde{y} \le 0$, rather than with $E \ln \tilde{y} < 0$. However, the distinction is nil if utility is differentiable.

or 3(*ii*). We first show that $R_u(x)$ is decreasing and convex, whenever $\hat{r}_u(\theta)$ is decreasing and convex. We then show how there is an isomorphic relationship between standard (absolute) risk aversion of the affiliated utility function $\hat{u}$ and standard relative risk aversion of *u*.

From equation (10), we see that

$$(12) \qquad R_u'(x) = \frac{1}{x}\ \hat{r}_u'(\ln x)$$

and

$$(13) \qquad R_u''(x) = \frac{1}{x^2}[\hat{r}_u''(\ln x) - \hat{r}_u'(\ln x)].$$

Consequently, since $x>0$, it follows from equation (12) that $R_u(x)$ is decreasing in x if and only if $\hat{r}_u(\ln x)$ is decreasing in $\ln x$. Moreover, if $\hat{r}_u$ is decreasing and convex, it follows from equation (13) that $R_u(x)$ is also convex. Hence, the conditions holding in Proposition 3(*i*) imply those of Proposition 1(*i*), except for $R_u > 1$.

The property of standard risk aversion, as presented in Kimball (1993), is analyzed at length in Gollier (2001). It is especially useful since it is easily characterized by decreasing absolute risk aversion and decreasing absolute prudence, where absolute prudence is measured as $p(x) = -\frac{u'''(x)}{u''(x)}$. If $u'''(x) > 0$, preferences are said to be prudent. If the affiliated utility function is standard risk averse, we may apply Proposition 3(*ii*) to conclude that *v* is more risk averse than *u*.

We first obtain a preliminary result that will prove useful. Straightforward calculations show that

$$(14) \qquad R_u'(x) = \frac{-u''(x)u'(x) - xu'''(x)u'(x) + x[u''(x)]^2}{[u'(x)]^2} = r_u(x)[1 - P_u(x) + R_u(x)],$$

where $P_u(x) \equiv \frac{-xu'''(x)}{u''(x)}$ denotes the measure of relative prudence. Consequently, we directly obtain the following result.

<u>Lemma 3</u>: *$R_u'(x) \gtreqless 0$ if and only if $P_u(x) \lesseqgtr 1 + R_u(x)$.*

We already know that $\hat{u}$ is risk averse whenever *u* is: that is, $\hat{u}$ inherits risk aversion from *u*. Unfortunately, the same cannot always be said for the property of prudence. Indeed, from the derivatives in (9), we can calculate relative prudence:

$$(15) \qquad P_u(x) \equiv \frac{-xu'''(x)}{u''(x)} = \frac{-\hat{u}'''(\ln x)}{\hat{u}''(\ln x)} + 1 \equiv \hat{p}_u(\ln x) + 1 .$$

The following results follow directly from (15):

<u>Lemma 4</u>: *Affiliated utility $\hat{u}$* is prudent if and only if relative prudence of *u* exceeds one.

<u>Lemma 5</u>: *If u exhibits decreasing relative risk aversion, the affiliated utility function $\hat{u}$ exhibits prudence.*

<u>Proof</u>: From Lemma 3, decreasing relative risk aversion of *u* implies that $P_u(x) > 1 + R_u(x)$. Since R_u is positive, the conclusion follows from Lemma 4. ■

We are now ready to prove that standard relative risk aversion of *u* is equivalent to standard (absolute) risk aversion of $\hat{u}$. This is precisely the grounds for our defining the affiliated utility function as we do in (8).

Lemma 6: The following two conditions are equivalent:

(*i*) $\hat{u}$ *exhibits standard risk aversion*

(*ii*) *u exhibits standard relative risk aversion (i.e. relative risk aversion and relative prudence that are both decreasing in wealth.)*

Proof: From (12), we know that $\hat{u}$ exhibits decreasing absolute risk aversion if and only if *u* exhibits decreasing relative risk aversion. From (15), it follows that

$$(16) \qquad P_u'(x) = \frac{1}{x} \ \hat{p}_u'(\ln x).$$

Hence, $\hat{u}$ exhibits decreasing absolute prudence if and only if *u* exhibits decreasing relative prudence as well. ■

From Lemma (6) and Proposition 3(*ii*), we obtain our main result of this section:

Proposition 4: Standard relative risk aversion of u is sufficient for v to be more risk averse than u.

Proposition 4 is a direct counterpart to the result of Eeckhoudt and Kimball (1992), who showed that standard risk aversion is sufficient to cause an individual to behave more cautiously in the presence of an additive background risk. Here we see that standard relative risk aversion plays the same role for the case where we introduce a multiplicative background risk.

Suppose once again that *u* belongs to the HARA class of utility functions, $u(x) = \xi(\eta + \frac{x}{\gamma})^{1-\gamma}$. Now $R_u(x) = \dfrac{x}{\eta + \frac{1}{\gamma}x}$. Thus, it follows easily that *u* exhibits decreasing relative risk aversion if and only if $\eta < 0$ and $\gamma > 0$. To see that *u* exhibits standard relative risk aversion, note that $P_u(x) = \frac{1+\gamma}{\gamma} R_u(x)$. Thus, u exhibits decreasing relative prudence if and only if u exhibits decreasing relative risk aversion. Thus we obtain the following useful result, which we list as a consequence of Proposition 4.

Corollary to Proposition 4: If utility is of the HARA class, then decreasing relative risk aversion of u is sufficient for v to be more risk averse than u.

6. Comparative Risk Aversion

We start here by examining some intrapersonal characteristics of risk aversion. We will later examine some interpersonal characteristics. From equation (4), we see trivially that $R_v(x)$ will be everywhere greater than [less than] one if $R_u(x)$ is everywhere greater than [less than] one. This result is more than just a technicality. Since many results in the literature on choice under uncertainty specify a global condition that either $R_u(x)>1$ or $R_u(x)<1$, such results also will hold in the presence of a multiplicative background risk, since $R_v(x)$ also will satisfy the appropriate property.

More generally, it follows trivially from equation (4) that

Proposition 5: Given any $\tilde{y} \in \Gamma(x)$, with distribution function G, $\inf\{R_u(xy)\} \le R_v(x) \le \sup\{R_u(xy)\} \quad \forall y \in Supp(G)$.

A key result in the literature on additive background risk is that the properties of constant absolute risk aversion and decreasing absolute risk aversion for utility are carried over to the derived utility function. On the other hand, the property of increasing absolute risk aversion does not always carry over. We next develop analogous results for relative risk aversion in the case of a multiplicative background risk. We have already seen that *v* inherits constant relative risk aversion from *u*. Indeed, the level of constant risk aversion is identical. To see that the same holds true for decreasing relative risk aversion, we first require the following Theorem, which is due to Gollier and Kimball (1996). A proof of this Theorem can also be found in Gollier (2001).

Lemma 7 (Diffidence Theorem, Gollier and Kimball): *Let Λ denote the set of all random variables with support contained in the interval $[a,b]$ and let f and g be two real-valued functions. The following two conditions are equivalent:*

(i) *For any* $\tilde{y} \in \Lambda, \quad Ef(\tilde{y}) = 0 \quad \Rightarrow \quad Eg(\tilde{y}) \geq 0.$

(ii) $\exists m \in \mathbb{R}$ *such that* $g(y) \geq mf(y) \quad \forall y \in [a,b].$

We now are ready to show that *v* also inherits decreasing relative risk aversion from *u*.[11]

Proposition 6: *Let $\tilde{y}$ have a bounded support. If u exhibits nonincreasing relative risk aversion, then so does the derived utility function v.*

Proof: It follows from Lemma 3, that we need to show that, $\forall x$,

$$(17) \qquad P_u(x) \geq 1 + R_u(x) \quad \Rightarrow \quad P_v(x) \geq 1 + R_v(x).$$

That is, we must show that

$$(18) \qquad \frac{-Eu'''(x\tilde{y})\tilde{y}^3 x}{Eu''(x\tilde{y})\tilde{y}^2} \geq \frac{-Eu''(x\tilde{y})\tilde{y}^2 x}{Eu'(x\tilde{y})\tilde{y}} + 1.$$

[11] Although aesthetically unappealing, the limitation to bounded supports is not particularly restrictive. We already limit $\tilde{x}$ and $\tilde{y}$ to be positive, so set *a*=0. Now, for any ε>0, we can always find a value for *b* such that the probability that $\tilde{y} > b$ is less than ε.

Inequality (18) is equivalent to the following:

(19) $$E[u''(x\tilde{y})\tilde{y}^2 x + (\lambda - 1)u'(x\tilde{y})\tilde{y}] = 0 \quad \Rightarrow \quad E[u'''(x\tilde{y})\tilde{y}^3 x + \lambda u''(x\tilde{y})\tilde{y}^2] \geq 0 .$$

By the Diffidence Theorem, (19) will hold if we can find a real number *m*, such that

(20) $$u'''(xy)y^3 x + \lambda u''(xy)y^2 \geq m[u''(xy)y^2 x + (\lambda - 1)u'(xy)y] \quad \forall y .$$

The left-hand side of (20) can be written as

(21) $$\frac{xyu''(xy)}{u'(xy)} \quad \frac{u'(xy)y}{x}\left[\frac{xyu'''(xy)}{u''(xy)} + \lambda\right] = -R_u(xy)\frac{u'(xy)y}{x}\left[\lambda - P_u(xy)\right].$$

Since $P_u(x) \geq 1 + R_u(x)$, it follows from (20) and (21) that

(22) $$u'''(xy)y^3 x + \lambda u''(xy)y^2 \geq -R_u(xy)\frac{u'(xy)y}{x}\left[\lambda - 1 - R_u(xy)\right].$$

From (20) and (22), we would be done if we could find an *m*, such that

(23) $$\begin{aligned} -R_u(xy)\frac{u'(xy)y}{x}\left[\lambda - 1 - R_u(xy)\right] &\geq m[u''(xy)y^2 x + (\lambda - 1)u'(xy)y] \\ &= mu'(xy)y[\lambda - 1 - R_u(xy)]. \end{aligned}$$

This follows by taking $m = (1 - \lambda)/x$, since we then obtain (23) is equivalent to

(24) $$-R_u(xy)\left[\lambda - 1 - R_u(xy)\right] + (\lambda - 1)[\lambda - 1 - R_u(xy)] = [\lambda - 1 - R_u(xy)]^2 \geq 0 .$$

Hence, (17) holds and *v* exhibits decreasing relative risk aversion. ■

We next turn to examining some interpersonal characteristics of comparative risk aversion. Kihlstrom, et al. (1981) and Ross (1981) examined these for the case of an additive background risk.[12] Their results are special cases of more general results found in Nachman (1982). Nachman is one of the few who considers the case of multiplicative background risks as a special case of his general results, albeit briefly. The basic question we address is the following: If agent 1 is more risk averse than agent 2, will this property be preserved in the presence of a multiplicative background risk? That is, if u_1 is more risk averse than u_2, when will it follow that v_1 is also more risk averse than v_2? One result that is quite easy to obtain is the following:

<u>Corollary to Proposition 5</u>: Let u^a and u^b be risk-averse utility functions such that u^a is more risk averse than u^b, i.e. $R_u^a(x) \geq R_u^b(x) \quad \forall x$. If $\exists \lambda \in \mathbb{R}$ such that $\forall x \quad R_u^a(x) \geq \lambda \geq R_u^b(x)$, then v^a is more risk averse than v^b.

<u>Proof</u>: Follows directly from Proposition 5 and equation (4). ■

The proof of the Corollary also follows directly from the following more general result, which is due to Nachman (1982). We include it here for completeness.

<u>Proposition (Nachman)</u>: Let u^a and u^b be risk-averse utility functions such that u^a is more risk averse than u^b, i.e. $R_u^a(x) \geq R_u^b(x) \;\; \forall x$. If there exists a function u^c such that $R_u^a(x) \geq R_u^c(x) \geq R_u^b(x) \;\; \forall x$ and $R_u^c(x)$ is nonincreasing, then v^a is more risk averse than v^b.

It follows easily from Nachman's result that v^a will be more risk averse than v^b if either of the utility functions, u^a or u^b, exhibits nonincreasing relative risk aversion.

[12] Actually, Ross considers the background risk to be mean-independent, which is not as restrictive as the assumption of independence.

This result is a direct counterpart to the result by Kihlstrom, et al. in the case of additive background risk.

7. Concluding Remarks

The notion that markets are complete is a mathematical nicety that does not hold true in practice. Many types of political, human-capital and social risks, as well as some financial risks, are not represented by direct contracts. Obviously, many of these risks can be hedged indirectly - - so-called “cross hedging.” However, even when such “background risks” are independent of other risks and cannot be “hedged” per se, they may have an impact upon risk-taking strategies that are within the control of the economic agent. Much has been done over the past twenty years in examining the effects of additive background risks. But surprisingly little has been done to systematically study economic decision making in the presence of a multiplicative background risk.

This paper is a first step towards developing a comprehensive theory of background risk in this direction. As the few examples in our introduction show, models with such multiplicative background risks are not hard to find within the literature. Whereas properties of absolute risk aversion play a key role in analyzing the effects of an additive background risk, properties of relative risk aversion are the most important in examining behavior in the presence of a multiplicative background risk. However, results for the case of a multiplicative background risk do not simply “mirror” those for the case where the background risk is additive. An understanding of the basic concepts presented here hopefully might help us understand a multitude of results for which standard theories (in the absence of any background risk) yield predictions that seem at odds with everyday observations of reality.

Since risk aversion captures all the essential information about preferences within an expected-utility framework, our focus here has been on comparing risk aversion with and without a multiplicative background risk. As we learn more about these inherent properties, we hopefully will be able to find better models to use in the realm of positive theories.

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